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A generalized Weierstrass elliptic function expansion method for solving some nonlinear partial differential equations

E.A. Saied, Reda G. Abd El-Rahman, Marwa I. Ghonamy *

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

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ABSTRACT

The present paper deals with families of non-trivial solutions of the equation $(\frac{\mathrm{d}}{\mathrm{d}\xi}w)^2 = P\,w^4(\xi) + Q\,w^2(\xi) + R$. On the basis of these solutions, a direct and generalized algebraic algorithm is described for constructing the new solutions of some nonlinear partial differential equations (NLPDEs). Subsequently, many new and more general exact solutions in terms of the Weierstrass elliptic function $\wp(\xi; g_2, g_3)$ are obtained. The method can be applied to other NLPDEs in mathematical physics.

1. Introduction

As seeking exact solutions of nonlinear physics equations is important and interesting, many powerful methods have been presented. Among these are Hirota's bilinear methods [1], the inverse scattering transform [1], Painlevé expansions [2], homogeneous balance method [3,4], tanh method [5,6], extended tanh method [7,8], modified extended tanh method [9], sine–cosine method [10–12], factorization method [13], and so on. It well known that many nonlinear evolution equations have been shown to possess periodic solutions. Recently, some methods have been presented for seeking periodic solutions, such as the F-expansion method [14,15]. The F-expansion method was later extended in different manners [16–18]. More recently, there has been available a new algebraic method for seeking exact solitary wave solutions of NLPDEs, which can be expressed as polynomials in an element that satisfies a more general ordinary differential equation such as the Riccati equation [19] ($w_\xi = b_0 + w^2$), the first-kind elliptic equation [20] ($w_\xi^2 = b_0 + b_2 w^2 + b_4 w^4$), the general elliptic equation [21] ($w_\xi^2 = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4$), etc.

In Ref. [22] Huber used the solutions of ($w_\xi^2 = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4$) in terms of Weierstrass elliptic functions; this thus led to the calculation of a new class of solutions. Also, we know that the Weierstrass elliptic function can be written in terms of the Jacobi elliptic function, and the hyperbolic and trigonometric functions are just special cases of Jacobi elliptic functions under certain conditions.

In this paper, we will attempt to develop an algorithm in terms of the Weierstrass elliptic function $\wp(\xi; g_2, g_3)$ [22] in order to seek for doubly periodic solutions of new types for nonlinear wave equations in mathematical physics. The method is simply called the generalized Weierstrass elliptic function expansion method and may be performed using a computer with the aid of symbolic computation. Moreover, we have applied it to some nonlinear wave equations. Here we will choose seven nonlinear wave equations to illustrate the algorithm. The results obtained show that the algorithm is more powerful for seeking doubly periodic solutions of nonlinear wave equations.

* Corresponding author.
E-mail address: marwamath@yahoo.com (M.I. Ghonamy).

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Also, by using a simple transformation, we have shown that the Korteweg–de Vries (KdV) equation, the (3 + 1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (2 + 1)-dimensional Boussinesq equation, the modified KdV equation, the (2 + 1)-dimensional modified KP equation, the nonlinear wave equation, the generalized Davey–Stewartson (GDS) equations, the Davey–Stewartson (DS) equations, and the generalized Zakharov (GZ) equations can be reduced to one of the following equations:

$$u''(\xi) + h_1 u(\xi) + h_2 u^2(\xi) = 0 \tag{1.1}$$

or

$$u''(\xi) + k_1 u(\xi) + k_3 u^3(\xi) = 0 \tag{1.2}$$

where h_1, h_2, k_1, k_3 are arbitrary constants.

The paper is organized as follows: In Section 2, first we briefly give the steps of the method. In Sections 3 and 4, we apply the method to solve Eqs. (1.1) and (1.2). In Section 5, by using the results obtained in Sections 3 and 4, the corresponding solutions of the KdV equation, (3 + 1)-dimensional KP equation, (2 + 1)-dimensional Boussinesq equation, modified KdV equation, (2 + 1)-dimensional modified KP equation, nonlinear wave equation, GDS equations, DS equations, and GZ equations are obtained.

2. Method and its algorithm

The main idea of this method is to use the solutions $w(\xi)$ which satisfy the first-order equation

$$w'(\xi) = \sqrt{P w^4(\xi) + Q w^2(\xi) + R}, \tag{2.1}$$

where $w' = \frac{\mathrm{d}w}{\mathrm{d}\xi}$, $\xi = \xi(x, y, z, t)$ and P, Q, R are real parameters. The solutions $w(\xi)$ can be expressed using the Weierstrass elliptic function $\wp(\xi; g_2, g_3)$ satisfying a nonlinear ordinary differential equation

$$(\wp(\xi))'^2 = 4\wp(\xi)^3 - g_2\wp(\xi) - g_3, \tag{2.2}$$

where g_2, g_3 are real parameters, called invariants [23,24], which has another equivalent form:

$$(\wp(\xi))'' = 6\wp(\xi)^2 - \frac{1}{2}g_2, \tag{2.3}$$

and we know that Eq. (2.1) possesses the following solution:

$$w_1(\xi) = \sqrt{\frac{1}{P}\left(\wp(\xi; g_2, g_3) - \frac{Q}{3}\right)}, \tag{2.4}$$

$$w_2(\xi) = \sqrt{\frac{3R}{3\wp(\xi; g_2, g_3) - Q}}, \tag{2.5}$$

where

$$g_2 = \frac{4Q^2 - 12RP}{3} \quad \text{and} \quad g_3 = \frac{4Q(9PR - 2Q^2)}{27}. \tag{2.6}$$

$$w_3(\xi) = \sqrt{\frac{12R\wp(\xi; g_2, g_3) + 2R(2Q + D_{1,2})}{12\wp(\xi; g_2, g_3) + D_{1,2}}}, \tag{2.7}$$

where

$$\begin{aligned} D_{1,2} &= \frac{-5Q \pm \sqrt{9Q^2 - 36RP}}{2}, \\ g_2 &= -\frac{1}{12}(5QD_{1,2} + 4Q^2 + 33PQR), \\ g_3 &= -\frac{1}{216}(-21Q^2D_{1,2} - 20Q^3 + 63RPD_{1,2} + 27PQR). \end{aligned} \tag{2.8}$$

$$w_4(\xi) = \frac{6\sqrt{R}\wp(\xi; g_2, g_3) + Q\sqrt{R}}{3\wp'(\xi; g_2, g_3)}, \tag{2.9}$$

$$w_5(\xi) = \frac{3\sqrt{\frac{1}{P}}\wp'(\xi; g_2, g_3)}{6\wp(\xi; g_2, g_3) + Q}, \tag{2.10}$$

where

$$g_2 = \frac{1}{12}(Q^2 + 12RP), \qquad g_3 = \frac{1}{216}(36RQP - Q^3). \tag{2.11}$$

$$w_6(\xi) = \frac{\sqrt{\frac{-15Q}{2P}}\,\wp(\xi; g_2, g_3)}{3\wp(\xi; g_2, g_3) + Q}, \tag{2.12}$$

where

$$R = \frac{5Q^2}{36P}, \qquad g_2 = \frac{2Q^2}{9}, \qquad g_3 = \frac{Q^3}{54}. \tag{2.13}$$

For a given PDE with $u(x, y, z, t)$ in four independent variables x, y, z, t,

$$H(u, u_t, u_x, u_y, u_z, u_{xx}, u_{xt}, u_{yy}, \ldots) = 0, \tag{2.14}$$

the travelling wave solution, $u(x, y, z, t) = u(\xi)$, $\xi = \alpha x + \beta y + \gamma z - \lambda t$, reduces (2.14) to a nonlinear ordinary differential equation

$$G(u, u', u'', \ldots) = 0. \tag{2.15}$$

Here the prime denotes $\mathrm{d}/\mathrm{d}\xi$. We assume that solutions of Eq. (2.15) can be expressed in the form

$$u = a_0 + \sum_{i=1}^{n} a_i w^i(\xi) + b_i w^{-i}(\xi) + c_i w^{i-1}(\xi) w'(\xi) + d_i w'(\xi) w^{-i}(\xi), \tag{2.16}$$

where $n \neq 0$, a_0, a_i, b_i, c_i, d_i are parameters to be determined later.

To determine n, we define a polynomial degree function as $D[u] = n$, and thus we have $D\left[\frac{\mathrm{d}^s u}{\mathrm{d}\xi^s}\right] = n + s$, $D\left[u^\alpha \left(\frac{\mathrm{d}^s u}{\mathrm{d}\xi^s}\right)^\gamma\right] = n\alpha + \gamma(n + s)$. Therefore we can determine n in (2.16) by balancing the highest degree linear term and nonlinear terms in (2.14) or (2.15). Note that if $n = 0$, then the method does not work.

The process requires the following steps:

Step 1. Determine n in Eq. (2.16) by balancing the linear term of highest order with the nonlinear term in Eq. (2.15). (If $n \neq 0$ is not a positive integer, then we firstly make the transformation $u = v^n$, and then we carry out this step again.)

Step 2. Substitute (2.16) with the known parameter n into the left side of the obtained ODE (2.15) along with (2.1), and get an expression. And then take the numerator of the expression to get a polynomial equation for $w'^i w^j$ $(i = 0, 1; j = 0, 1, 2, 3, \ldots)$. Set to zero the coefficients of the polynomial obtained to get a set of algebraic equations with respect to the unknowns $\alpha, \beta, \gamma, \lambda, P, Q, R, a_0, a_i, b_i, c_i, d_i$ $(i = 1, \ldots, n)$.

Step 3. Solving these equations by use of Mathematica, we will obtain the explicit expressions for a_0, a_i, b_i, c_i, d_i $(i = 1, \ldots, n)$ and ξ. Finally, substituting these results into Eq. (2.16) and using special solutions of Eq. (2.1) gives the general form of the travelling wave solutions.

3. Weierstrass elliptic function solutions of (1.1)

Now let us apply the method of Section 2 to Eq. (1.1). From balancing we get that $n = 2$, so the solution of the NLODE (1.1) is of the form

$$u = a_0 + a_1 w(\xi) + a_2 w^2(\xi) + \frac{b_1}{w(\xi)} + \frac{b_2}{w^2(\xi)} + c_1 w'(\xi) + c_2 w(\xi) w'(\xi) + \frac{d_1 w'(\xi)}{w(\xi)} + \frac{d_2 w'(\xi)}{w^2(\xi)}, \tag{3.1}$$

where a_0, a_i, b_i, c_i, d_i $(i = 1, 2)$ are constants to be determined, $w(\xi)$ satisfies ODE (2.1). Substituting (3.1) into (1.1) along with (2.1) and collecting the coefficients of the $w'^i w^j$ $(i = 0, 1; j = 0, 1, 2, 3, \ldots)$ we have a set of algebraic equations which are solved to get the following cases:

Case 1.

$$\begin{aligned} &a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = d_2 = 0, \qquad b_2 = \frac{-6R}{h_2}, \\ &a_0 = \frac{-h_1 - 4Q}{2h_2}, \qquad P = \frac{-{h_1}^2 + 16Q^2}{48R}. \end{aligned} \tag{3.2}$$

Case 2.

$$\begin{aligned} &a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0, \qquad a_2 = \frac{-6P}{h_2}, \\ &a_0 = \frac{-h_1 - 4Q}{2h_2}, \qquad P = \frac{-{h_1}^2 + 16Q^2}{48R}. \end{aligned} \tag{3.3}$$

Case 3.

$$
\begin{aligned}
&a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = 0, \qquad a_2 = \frac{-6P}{h_2}, \\
&a_0 = \frac{-h_1 - 4Q}{2h_2}, \qquad P = \frac{{h_1}^2 - 16Q^2}{192R}.
\end{aligned} \tag{3.4}
$$

Case 4.

$$
\begin{aligned}
&a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = 0, \qquad b_2 = \frac{-3R}{h_2}, \\
&d_2 = \frac{3\sqrt{R}}{h_2}, \qquad a_0 = \frac{-h_1 - Q}{2h_2}, \qquad P = \frac{{h_1}^2 - Q^2}{12R}.
\end{aligned} \tag{3.5}
$$

Case 5.

$$
\begin{aligned}
&a_1 = b_1 = b_2 = c_2 = d_1 = d_2 = 0, \qquad a_2 = \frac{-3P}{h_2}, \\
&c_1 = \frac{3\sqrt{P}}{h_2}, \qquad a_0 = \frac{-h_1 - Q}{2h_2}, \qquad P = \frac{{h_1}^2 - Q^2}{12R}.
\end{aligned} \tag{3.6}
$$

Case 6.

$$
\begin{aligned}
&a_1 = b_1 = c_2 = d_1 = 0, \qquad a_2 = \frac{-3P}{h_2}, \qquad b_2 = \frac{-3R}{h_2}, \qquad c_1 = \frac{3\sqrt{P}}{h_2}, \\
&d_2 = \frac{3\sqrt{R}}{h_2}, \qquad a_0 = \frac{-h_1 - Q + 6\sqrt{P}\sqrt{R}}{2h_2}, \qquad Q = -30\sqrt{P}\sqrt{R} + \sqrt{{h_1}^2 + 768PR}.
\end{aligned} \tag{3.7}
$$

Therefore, we get the following solutions of (3.1):

$$
u_1 = \frac{-h_1 - 4Q}{2h_2} - \frac{6R}{h_2 w^2(\xi)}, \tag{3.8}
$$

$$
u_2 = \frac{-h_1 - 4Q}{2h_2} - \frac{\left(-{h_1}^2 + 16Q^2\right) w^2(\xi)}{8Rh_2}, \tag{3.9}
$$

$$
u_3 = \frac{-h_1 - 4Q}{2h_2} - \frac{6R}{h_2 w^2(\xi)} - \frac{\left({h_1}^2 - 16Q^2\right) w^2(\xi)}{32Rh_2}, \tag{3.10}
$$

$$
u_4 = \frac{-h_1 - Q}{2h_2} - \frac{3R}{h_2 w^2(\xi)} + \frac{3\sqrt{R}w'(\xi)}{h_2 w^2(\xi)}, \tag{3.11}
$$

$$
u_5 = \frac{-h_1 - Q}{2h_2} - \frac{\left({h_1}^2 - Q^2\right) w^2(\xi)}{4Rh_2} + \frac{\sqrt{3}\sqrt{\frac{{h_1}^2 - Q^2}{R}}\, w'(\xi)}{2h_2}, \tag{3.12}
$$

$$
u_6 = \frac{-h_1 + 36\sqrt{P}\sqrt{R} - \sqrt{{h_1}^2 + 768PR}}{2h_2} - \frac{3R}{h_2 w^2(\xi)} - \frac{3Pw^2(\xi)}{h_2} + \frac{3\sqrt{P}w'(\xi)}{h_2} + \frac{3\sqrt{R}w'(\xi)}{h_2 w^2(\xi)}, \tag{3.13}
$$

where $w(\xi)$ is one of $w_1(\xi), w_2(\xi), \ldots, w_6(\xi)$, which are given in Eqs. (2.4)–(2.13).

4. Weierstrass elliptic function solutions of (1.2)

We seek the solutions of Eq. (1.2) in the new and more general form by applying the previous method. From balancing we get that $n = 1$; then the solutions become

$$
u = a_0 + a_1 w(\xi) + b_1 w^{-1}(\xi) + c_1 w'(\xi) + d_1 w'(\xi) w^{-1}(\xi), \tag{4.1}
$$

where a_0, a_1, b_1, c_1 and d_1 are constants to be determined. According to the above mentioned steps in Section 2, the following solutions are found for (1.2):

$$
u_1 = \frac{\sqrt{2}\sqrt{P}w(\xi)}{\sqrt{-k_3}}, \qquad Q = -k_1, \tag{4.2}
$$

$$
u_2 = \frac{\sqrt{2}w'(\xi)}{\sqrt{-k_3}w(\xi)}, \qquad Q = \frac{k_1}{2}, \tag{4.3}
$$

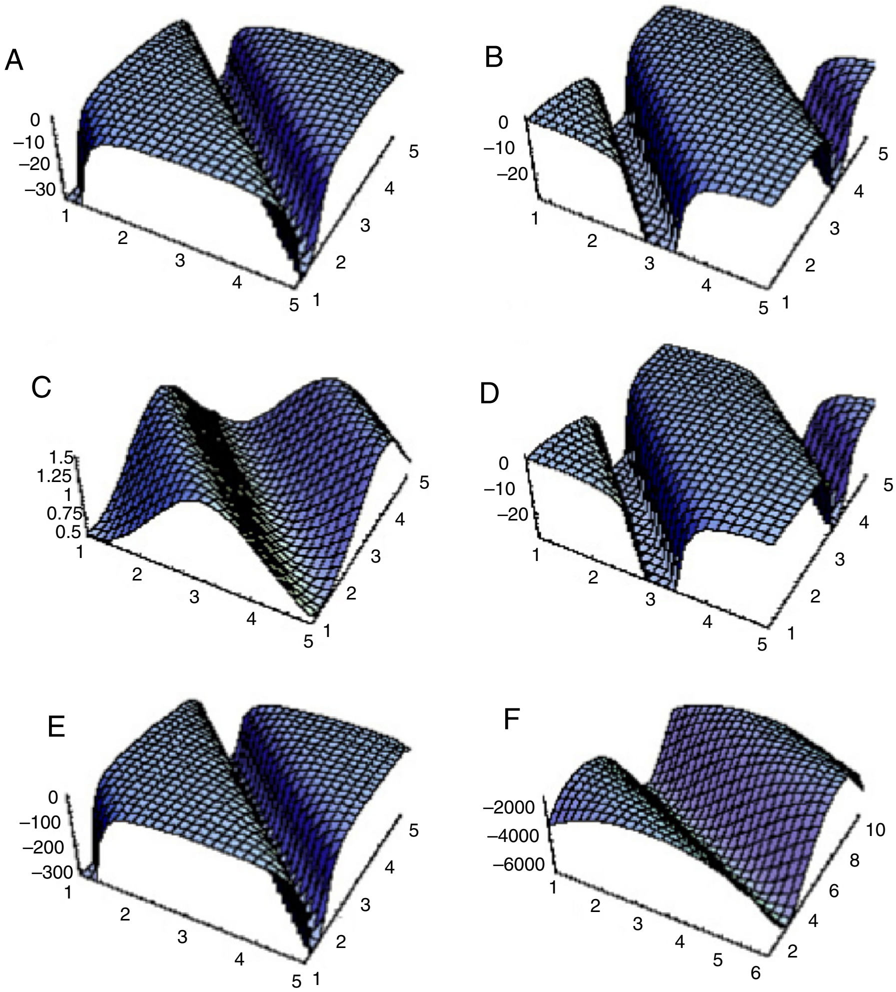

Fig. 1. The surfaces show the solution u_1 of Eq. (1.1) at $h_1 = -1$, $h_2 = 3$ with (A) $w(\zeta) = w_1(\zeta)$, (B) $w(\zeta) = w_2(\zeta)$, (C) $w(\zeta) = w_3(\zeta)$, (D) $w(\zeta) = w_4(\zeta)$, (E) $w(\zeta) = w_5(\zeta)$, (F) $w(\zeta) = w_6(\zeta)$.

$$u_3 = \frac{\sqrt{2}\left(\sqrt{R} + \sqrt{P}w^2(\xi)\right)}{\sqrt{-k_3}w(\xi)}, \qquad Q = 6\sqrt{P}\sqrt{R} - k_1, \tag{4.4}$$

$$u_4 = \frac{\sqrt{2R}}{\sqrt{-k_3}w(\xi)}, \qquad Q = -k_1, \tag{4.5}$$

$$u_5 = \frac{-\sqrt{R} - \sqrt{P}w^2(\xi) - w'(\xi)}{\sqrt{-2k_3}w(\xi)}, \qquad Q = -6\sqrt{PR} + 2k_1, \tag{4.6}$$

where $w(\xi)$ is one of $w_1(\xi), w_2(\xi), \ldots, w_6(\xi)$, which are given in Eqs. (2.4)–(2.13).

For different forms of the functions $w(\xi)$, we plot the solution u_1 of Eqs. (1.1) and (1.2) as an example to show the behaviors of its solutions (Figs. 1 and 2).

5. Exact solutions for some class of NLPDEs

In this section, by using the results obtained in the preceding sections, we will construct the corresponding solutions of the KdV equation, (3 + 1)-dimensional KP equation, (2 + 1)-dimensional Boussinesq equation, modified KdV equation, (2 + 1)-dimensional modified KP equation, nonlinear wave equation, GDS equations, DS equations, and GZ equations.

5.1. KdV equation

Let us consider the KdV equation [25]

$$u_t = u_{xxx} + 6uu_x, \tag{5.1}$$

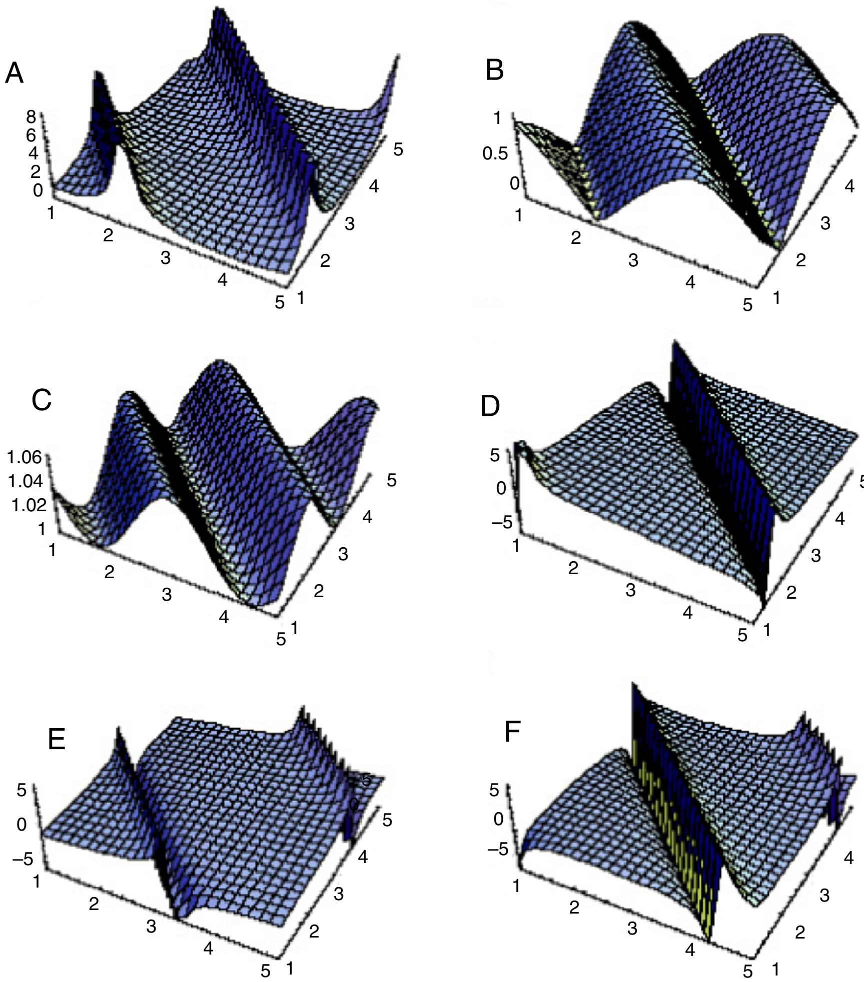

Fig. 2. The surfaces show the solution u_1 of Eq. (1.2) at $k_1 = 1$, $k_3 = -2$ with (A) $w(\zeta) = w_1(\zeta)$, (B) $w(\zeta) = w_2(\zeta)$, (C) $w(\zeta) = w_3(\zeta)$, (D) $w(\zeta) = w_4(\zeta)$, (E) $w(\zeta) = w_5(\zeta)$, (F) $w(\zeta) = w_6(\zeta)$.

We make the following formal travelling wave transformation:

$$u(x, t) = u(\xi), \quad \xi = \beta x - \lambda t, \tag{5.2}$$

where β, λ are constants to be determined. Substituting (5.2) into (5.1) gives

$$u''(\xi) + h_1 u(\xi) + h_2 u^2(\xi) = 0, \quad h_1 = \frac{\lambda}{\beta^3},\ h_2 = \frac{3}{\beta^2}. \tag{5.3}$$

Eq. (5.3) coincides with Eq. (1.1). Then the solutions of (5.1) are given by Eqs. (3.8)–(3.13).

5.2. (3 + 1)-*dimensional KP equation*

Let us now consider the (3 + 1)-dimensional KP equation [26]

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{5.4}$$

We make the following formal travelling wave transformation:

$$u(x, y, z, t) = u(\xi), \quad \xi = \alpha x + \beta y + \gamma z - \lambda t, \tag{5.5}$$

where α, β, γ, λ are constants to be determined. Substituting (5.5) into (5.4) gives

$$u''(\xi) + h_1 u(\xi) + h_2 u^2(\xi) = 0, \quad h_1 = \frac{(\beta^2 + \gamma^2 + \alpha\lambda)}{\alpha^4},\ h_2 = \frac{-3}{\alpha^2}, \tag{5.6}$$

where $u(\xi)$ is given in Eqs. (3.8)–(3.13).

5.3. (2 + 1)-dimensional Boussinesq equation

Let us consider a (2 + 1)-dimensional generalization of the Boussinesq equation [27]:

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0. \tag{5.7}$$

We make the following formal travelling wave transformation:

$$u(x, y, z, t) = u(\xi), \quad \xi = \alpha x + \beta y - \lambda t, \tag{5.8}$$

where α, β, λ are constants to be determined. Substituting (5.8) into (5.7) gives

$$u''(\xi) + h_1 u(\xi) + h_2 u^2(\xi) = 0, \quad h_1 = \frac{(\alpha^2 + \beta^2 - \lambda^2)}{\alpha^4},\ h_2 = \frac{1}{\alpha^2}. \tag{5.9}$$

It is easy to see that Eq. (5.9) coincides with Eq. (1.1). Then solutions of Eq. (5.7) are defined in (3.8)–(3.13).

5.4. Modified KdV equation

Let us consider the modified KdV equation [25]

$$u_t = u_{xxx} - 6u^2 u_x. \tag{5.10}$$

We make the following formal travelling wave transformation:

$$u(x, t) = u(\xi), \quad \xi = \beta x - \lambda t, \tag{5.11}$$

where β, λ are constants to be determined. Substituting (5.11) into (5.10) gives

$$u''(\xi) + k_1 u(\xi) + k_3 u^3(\xi) = 0, \quad k_1 = \frac{\lambda}{\beta^3},\ k_3 = \frac{-2}{\beta^2}, \tag{5.12}$$

which has solutions (4.2)–(4.6).

The KdV and mKdV forms are related via Miura transformation so the classes of solutions of the two equations are linked. So the new class of solutions derived here in (4.2)–(4.6) are linked via Miura transformation.

5.5. (2 + 1)-dimensional modified KP equation

Similarly, for the (2 + 1)-dimensional modified KP equation [28]

$$(u_t + 3\alpha u^2 u_x + u_{xxx})_x + u_{yy} = 0, \tag{5.13}$$

take

$$u(x, t) = u(\xi), \quad \xi = \beta x + \gamma y - \lambda t, \tag{5.14}$$

where β, γ, λ are constants to be determined. Substituting (5.14) into (5.13) gives

$$u''(\xi) + k_1 u(\xi) + k_3 u^3(\xi) = 0, \quad k_1 = \frac{\gamma^2 - \beta\lambda}{\beta^4},\ k_3 = \frac{\alpha}{\beta^2}. \tag{5.15}$$

Eq. (5.15) coincides with Eq. (1.2), where $u(\xi)$ is given by relations (4.2)–(4.6).

5.6. A nonlinear wave equation

Consider the nonlinear wave equation [29]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \tag{5.16}$$

where α, β and γ are constants. This equation contains some particularly important equations such as the Duffing, Klein–Gordon and Landau–Ginzburg–Higgs equations. We assume that Eq. (5.16) has an exact solution in the form

$$u(x, t) = u(\xi), \quad \xi = px - wt. \tag{5.17}$$

Substituting Eq. (5.17) into Eq. (5.16), we have

$$u''(\xi) + k_1 u(\xi) + k_3 u^3(\xi) = 0, \quad k_1 = \frac{\beta}{w^2 + \alpha p^2},\ k_3 = \frac{\gamma}{w^2 + \alpha p^2}, \tag{5.18}$$

which has solutions (4.2)–(4.6).

5.7. GDS, DS and GZ equations

We consider a class of NLPDEs with constant coefficients [30]

$$\mathrm{i}u_t + \mu(u_{xx} + D_1 u_{yy}) + E_1|u|^2 u + C_1 un = 0, \tag{5.19}$$

$$D_2 n_{tt} + (n_{xx} - E_2 n_{yy}) + C_2(|u|^2)_{xx} = 0, \tag{5.20}$$

where μ, D_i, E_i, C_i $(i = 1, 2)$ are real constants and $\mu \neq 0, D_1 \neq 0, C_1 \neq 0, C_2 \neq 0$. Eqs. (5.19) and (5.20) are a class of physically important equations. In fact, if one takes

$$\begin{aligned} &\mu = \frac{1}{2}K^2, \quad D_1 = 2\mu, \quad E_1 = \alpha, \quad C_1 = -1,\\ &D_2 = 0, \quad E_2 = D_1, \quad C_2 = -2\alpha, \quad K^2 = \pm 1, \end{aligned} \tag{5.21}$$

then Eqs. (5.19) and (5.20) represent the Davey–Stewartson (DS) equations [31]

$$\mathrm{i}u_t + \frac{1}{2}K^2(u_{xx} + K^2 u_{yy}) + \alpha|u|^2 u - un = 0, \tag{5.22}$$

$$n_{xx} - K^2 n_{yy} - 2\alpha(|u|^2)_{xx} = 0. \tag{5.23}$$

Also, if one takes

$$\begin{aligned} &n = n(x, t), \quad \text{i.e., } n_y = 0, \quad \mu = 1, \quad D_1 = 0, \quad E_1 = -2\lambda,\\ &D_2 = -1, \quad E_2 = 0, \quad C_2 = -1, \quad C_1 = 2, \end{aligned} \tag{5.24}$$

then Eqs. (5.19) and (5.20) become the generalized Zakharov (GZ) equations [32]

$$\mathrm{i}u_t + u_{xx} - 2\lambda|u|^2 u + 2un = 0, \tag{5.25}$$

$$n_{tt} - n_{xx} + (|u|^2)_{xx} = 0. \tag{5.26}$$

Since u is a complex function, we assume that

$$u(x, y, t) = \phi(\xi)\mathrm{e}^{\mathrm{i}(kx+ly-\Omega t)}, \qquad n(x, y, t) = n(\xi), \quad \xi = px + qy - rt, \tag{5.27}$$

where both $\phi(\xi)$ and $n(\xi)$ are real functions, k, l, p, q, Ω and r are constants to be determined later. Substituting Eq. (5.27) into Eqs. (5.19) and (5.20), we have the following ODE for $\phi(\xi)$ and $n(\xi)$:

$$\begin{aligned} &\mu(p^2 + D_1 q^2)\phi''(\xi) + [\Omega - \mu(k^2 + D_1 l^2)]\phi(\xi) + E_1\phi^3(\xi)\\ &\qquad + \mathrm{i}[-r + 2\mu(kp + D_1 lq)]\phi'(\xi) + C_1\phi(\xi)n(\xi) = 0, \end{aligned} \tag{5.28}$$

$$(D_2 r^2 + p^2 - E_2 q^2)n''(\xi) + C_2 p^2(\phi^2(\xi))'' = 0. \tag{5.29}$$

If we set

$$r = 2\mu(kp + D_1 lq), \tag{5.30}$$

then (5.28) and (5.29) reduce to

$$\mu(p^2 + D_1 q^2)\phi''(\xi) + [\Omega - \mu(k^2 + D_1 l^2)]\phi(\xi) + E_1\phi^3(\xi) + C_1\phi(\xi)n(\xi) = 0, \tag{5.31}$$

$$(D_2 r^2 + p^2 - E_2 q^2)n''(\xi) + C_2 p^2(\phi^2(\xi))'' = 0. \tag{5.32}$$

Integrating (5.32) once, we get

$$(D_2 r^2 + p^2 - E_2 q^2)n'(\xi) + C_2 p^2(\phi^2(\xi))' = \c{C}, \tag{5.33}$$

where Ç is an integration constant; then we take Ç $= 0$ and integrating the formula once again, we have

$$n(\xi) = \frac{C}{D_2 r^2 + p^2 - E_2 q^2} - \frac{C_2 p^2}{D_2 r^2 + p^2 - E_2 q^2}\phi^2(\xi). \tag{5.34}$$

Substituting (5.34) into (5.31) yields

$$\phi''(\xi) + k_1\phi(\xi) + k_3\phi^3(\xi) = 0, \tag{5.35}$$

where k_1, k_3 are defined by

$$\begin{aligned} &k_1 = \frac{B}{A}, \quad k_3 = \frac{D}{A}, \quad A = \mu(p^2 + D_1 q^2)(D_2 r^2 + p^2 - E_2 q^2),\\ &B = [C_1 C - (D_2 r^2 + p^2 - E_2 q^2)(\Omega - \mu(k^2 + D_1 l^2))],\\ &D = [E_1(D_2 r^2 + p^2 - E_2 q^2) - C_1 C_2 p^2]. \end{aligned} \tag{5.36}$$

Then the solutions of Eqs. (5.19) and (5.20) are

$$\begin{aligned} &u(x,y,t)=\phi(\xi)\mathrm{e}^{\mathrm{i}(kx+ly-\Omega t)},\\ &n(x,y,t)=\frac{C}{D_2r^2+p^2-E_2q^2}-\frac{C_2p^2}{D_2r^2+p^2-E_2q^2}\phi^2(\xi),\\ &r=2\mu(kp+D_1lq). \end{aligned} \tag{5.37}$$

The expressions for $\phi(\xi)$ appearing in these solutions are given in Section 4, k_1, k_3 are given by (5.36) and $\xi=px+qy-rt$. Then the solutions of DS equations (5.22) and (5.23) for the case

$$r=K^2(kp+K^2lq), \tag{5.38}$$

are

$$n(x,y,t)=\frac{C}{p^2-K^2q^2}+\frac{2\alpha p^2}{p^2-K^2q^2}\phi^2(\xi), \qquad u(x,y,t)=\phi(\xi)\mathrm{e}^{\mathrm{i}(kx+ly-\Omega t)}, \tag{5.39}$$

where $\phi(\xi)$ satisfies

$$\phi''(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0, \tag{5.40}$$

and k_1, k_3 are defined as follows:

$$\begin{aligned} &k_1=\frac{B}{A}, \qquad k_3=\frac{D}{A}, \qquad A=\frac{1}{2}K^2(p^2+K^2q^2)(p^2-K^2q^2),\\ &B=2C+(p^2-K^2q^2)(-2\Omega+K^2(k^2+K^2l^2)),\\ &D=-\alpha(p^2+K^2q^2), \qquad K^2=\pm 1. \end{aligned} \tag{5.41}$$

The expressions for $\phi(\xi)$ appearing in these solutions are given in Section 4 Eqs. (4.2)–(4.6), k_1, k_3 are given by (5.41) and $\xi=px+qy-rt$. Finally, the solutions of the GZ equations (5.25) and (5.26) for the case

$$r=2kp, \tag{5.42}$$

are

$$n(x,t)=\frac{C}{p^2-r^2}+\frac{p^2}{p^2-r^2}\phi^2(\xi), \qquad u(x,t)=\phi(\xi)\mathrm{e}^{\mathrm{i}(kx-\Omega t)}, \tag{5.43}$$

where $\phi(\xi)$ satisfies

$$\phi''(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0, \tag{5.44}$$

and k_1, k_3 are defined as follows:

$$\begin{aligned} &k_1=\frac{B}{A}, \qquad k_3=\frac{D}{A}, \qquad A=p^2(p^2-r^2),\\ &B=2C-(p^2-r^2)(\Omega-k^2),\\ &D=2(p^2-\lambda(p^2-r^2)). \end{aligned} \tag{5.45}$$

The expressions for $\phi(\xi)$ appearing in these solutions are given in Section 4 Eqs. (4.2)–(4.6), k_1, k_3 are given by (5.45) and $\xi=px-rt$.

Some basic properties of the limited behavior of class of solutions can be realized from Figs. 3a and 3b.

Remark 1. It is easy to see that the ansatz solution (2.16) is more general than the ansatz (3) constructed by Huber in [22], so it can be used to obtain more general solutions in terms of Weierstrass elliptic function $\wp(\xi;g_2,g_3)$. With the aid of Mathematica, we have verified all the solutions obtained in this paper by putting them back into the original NLPDEs (the KdV equation, the (3 + 1)-dimensional KP equation, the (2 + 1)-dimensional Boussinesq equation, the modified KdV equation, the (2 + 1)-dimensional modified KP equation, the nonlinear wave equation, the generalized Davey–Stewartson (GDS) equations, the Davey–Stewartson (DS) equations, the generalized Zakharov (GZ) equations).

Remark 2. The algorithm mentioned above succeeds with any NLPDEs reduced by travelling wave transformation to ODEs of types (1.1) or (1.2). The NLPDEs which cannot transform to Eq. (1.1) or (1.2) may be need some restriction to ensure the success of this algorithm.

For example the generalized Kuramoto–Sivashinsky equation [33]

$$\phi_t+\phi\nu\phi_{xxxx}+b\phi_{xxx}+\mu\phi_{xx}+\phi\phi_x=0,$$

with real parameters b, μ and $\nu\neq 0$, has Weierstrass solutions if we put $b^2-16\mu\nu=0$.

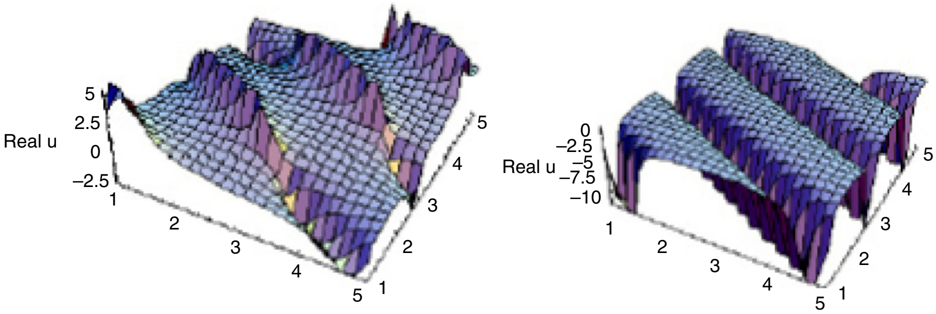

Fig. 3a. The surface shows the real part of solutions (5.43) with $\phi(\zeta) = u_1$, $w = w_1$, $p = 1$, $r = -2$, $\Omega = -1.\lambda = 2/3$, $k_1 = 1$, $k_3 = -2$, $P = 1$, $Q = -1$, $R = 1$, $k = -1$, $A = -3$, $B = -3$, $D = 6$, $C = 3/2$.

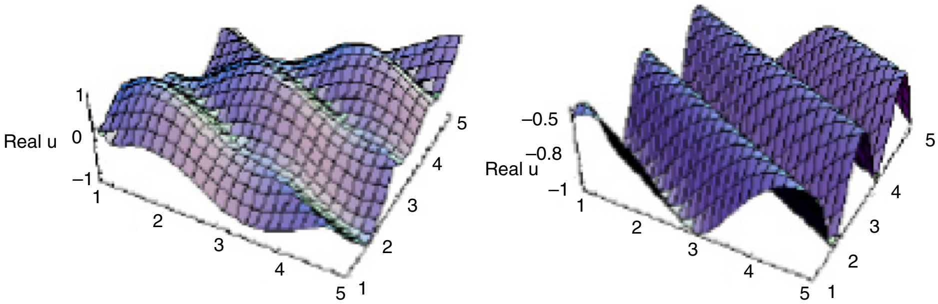

Fig. 3b. The surface shows the real part of solutions (5.43) with $\phi(\zeta) = u_1$, $w = w_2$, $p = 1$, $r = -2$, $\Omega = -1$, $\lambda = 2/3$, $k_1 = 1$, $k_3 = -2$, $P = 1$, $Q = -1$, $R = 1$, $k = -1$, $A = -3$, $B = -3$, $D = 6$, $C = 3/2$.

6. Conclusion

In this paper, we have proposed a generalized Weierstrass elliptic function expansion method for constructing more general exact solutions of NLPDEs. The advantage of the method is that it can be used to obtain more general exact solutions which cannot be obtained by the known Weierstrass elliptic function expansion methods (see, for example, Refs. [22,34]). We have also studied the behavior of some special solutions by plotting their figures.

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